JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **61**, 607–619 (1977)

Contractivity of C. Neumann's Operator in Potential Theory

JOSEF KRÁL AND IVAN NETUKA

Mathematical Institute of the Academy, Žitná 25, 11567 Praha 1, Czechoslovakia, and Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 18600 Praha 8, Czechoslovakia

Submitted by C. L. Dolph

More than 100 years ago Carl Neumann proposed a method for solving the Dirichlet problem on convex sets. His method of the arithmetic mean (cf. [9–11]) based on double layer potentials played an important role in potential theory. The Dirichlet principle (connected with the assumption that the Dirichlet integral always attains the admissible minimum), as employed by Riemann, was no longer considered reliable after Weierstrass' criticism. The method of the arithmetic mean, on the contrary, was appreciated as a correct method for obtaining the solution of the Dirichlet problem, although only on domains bounded by convex surfaces. Neumann's method delivered not only an existence theorem but also the expansion of the solution in a series suitable for approximate calculation. It stimulated further research and progress in the theory of integral equations (cf. [13]; the corresponding historical comments may also be found in [3, Chap. XI, Introduction]). Some 50 years later Lebesgue published a critical study [6] (reprinted in [7, pp. 107–122)] in which he pointed out a gap in the original proof of Neumann [10].[1] The error consisted in the assumption that a certain variable quantity (depending on the position of two points in the boundary and the decomposition of the boundary into two parts) must actually assume its maximal value. Lebesgue proved the invalidity of this reasoning; he writes [7, p. 112]: "Le raisonnement de Neumann, destiné à remplacer celui de Riemann, est donc fondé exactement sur la même confusion entre borne supérieure et maximum, justement critiquée par Weierstrass. Il est tout à fait étonnant que tout le monde, à commencer par Weierstrass, ait admis la validité du raisonnement de Neumann et que nos traités actuels continuent à opposer le raisonnement de Riemann, déclaré faux, à celui de Neumann, proclamé entièrement

[1] Lebesgue writes [7, p. 107]: "Carl Neumann . . . a donné pour la résolution du problème de Dirichlet une méthode restée justement célèbre; Neumann se bornait à l'étude des domaines convexes, Poincaré a justifié la méthode pour des cas étendus de domaines non convexes; les recherches de Fredholm ont fait mieux comprendre encore l'importance de cette méthode et les raisons de son succès. La critique que j'en veux faire ici ne portera que sur la légitimation classique pout le cas des domaines convexes."

ISSN 0022-247X

correct." We wish to point out here that Lebesgue's criticism was only partly justified. In another treatise [11] (published 10 years after that [10] criticized by Lebesgue), which apparently remained unknown to Lebesgue, Neumann admitted that his argument was not convincing. He writes in a footnote on p. 759: "Es könnte wohl sein, dass hier $\cdots$ die Tragweite der Weierstrass' schen Principen überschritten ist. Ich betrachte daher den hier $\cdots$ gegebenen Beweis nur als einen provisorischen, und werde denselben im folgenden § durch einen anderen und zwar absolut strengen Beweis ersetzen." The proof of Neumann's fundamental lemma for plane convex domains as given in [11, Sect. 6] is indeed quite correct and detailed. Neumann also investigated convex domains in 3-space and was fully aware of the exceptional role played by domains arising as an intersection of two cones with vertices in the boundary. In the later development Neumann's lemma retained its significance in connection with an approximate solution to boundary value problems for harmonic functions, constructing the Riemann mapping function for convex regions and related topics. Unfortunately, the correct proofs given by Neumann himself [11] and by Lebesgue [6] remained largely unnoticed and many texts presenting proofs of Neumann's lemma either contain gaps or introduce unnecessarily strong additional restrictions on the boundary; we refer the reader to Schober's paper [14] for the corresponding references and further comments.

The present paper has been inspired by a question occurring in Appendix A of the recent investigation of Kleinman and Wendland [4]. We introduce here Neumann's operator in full generality in the Euclidean space R^m of arbitrary dimension $m \geqslant 2$ and investigate necessary and sufficient conditions for its contractivity. In order to be able to formulate our main result we first introduce some

Notation. $\mathscr{H}_k$ denotes the k-dimensional Hausdorff measure in R^m (normalized in such a way that $\mathscr{H}_k(I^k) = 1$ for $I^k \subset R^m$ isometric with the unit cube $\langle 0, 1\rangle^k \subset R^k$); in particular, $\mathscr{H}_m$ reduces to the outer Lebesgue measure in R^m. We denote by $A = \mathscr{H}_{m-1}(\Gamma)$ the area of the unit sphere $\Gamma = \{\theta \in R^m; |\theta| = 1\}$. If $M \subset R^m$ and $z \in R^m$, then $\operatorname{contg}_z M$ stands for the contingent of M at z consisting of all the half-lines

$$H_z(\theta) = \{z + t\theta; t \geqslant 0\},$$

such that there is a sequence of points $z_n \in M \backslash \{z\}$ with

$$\lim_{n\to\infty} z_n = z \qquad \text{and} \qquad \lim_{n\to\infty}(z_n - z)/|z_n - z| = \theta.$$

The union of all the half-lines in $\operatorname{contg}_z M$ is denoted by

$$K_z(M) = \bigcup \operatorname{contg}_z M.$$

We put for $z \in R^m \backslash \{0\}$

$$\begin{aligned} p(z) &= 2[(m-2)A]^{-1} \mid z \mid^{2-m}, \qquad \text{if} \quad m > 2, \\ &= -2A^{-1} \log \mid z \mid, \qquad \text{if} \quad m = 2.^2 \end{aligned}$$

We fix an open set $G \subset R^m$ with a compact boundary $B \neq \varnothing$. $C^*(B)$ denotes the Banach space of all finite signed Borel measures with support in B; the norm $\| \mu \|$ of any $\mu \in C^*(B)$ is given by its total variation $\mid \mu \mid (R^m)$. With every $\mu \in C^*(B)$ we associate its potential

$$U\mu(l) = \int_{R^m} p(l-y)\, d\mu(y)$$

representing a harmonic function in G. An elementary calculation shows that, for any bounded Borel set $P \subset G$,

$$\int_P \mid \operatorname{grad} U\mu(x) \mid dx < \infty.$$

This makes it possible to introduce the functional $N^G U\mu$ over the space $\mathscr{D}$ of all infinitely differentiable functions φ with compact support in R^m by

$$\langle \varphi, N^G U\mu \rangle = \int_G \operatorname{grad} \varphi(x) \cdot \operatorname{grad} U\mu(x)\, dx.$$

It is easily seen that the distribution $N^G U\mu$ has support contained in B; $N^G U\mu$ was termed the generalized normal derivative of $U\mu$ (cf. [5], where further comments are given on pp. 512–513). We wish to mention here that already Plemelj [12] introduced a related concept termed "Strömung" as a convenient substitute for the classical normal derivative; see also [1].

If Φ is any distribution over $\mathscr{D}$ with support in B, we put

$$\| \Phi \| = \sup\{\langle \varphi, \Phi \rangle; \varphi \in \mathscr{D}, \mid \varphi \mid \leqslant 1\}.$$

It is well known that $\| \Phi \| < \infty$ is a necessary and sufficient condition for the existence of a $\mu \in C^*(B)$ representing Φ in the sense that

$$\langle \varphi, \Phi \rangle = \int_{R^m} \varphi\, d\mu, \qquad \varphi \in \mathscr{D};$$

such a μ is uniquely determined and is identified with Φ in the usual sense in distribution theory.

[2] For convenience we introduce here the additional multiplicative factor $2/A$ in comparison with the notation adopted in [5].

We are engaged with distributions

$$T_G{}^*\mu = N^G U\mu - \mu, \qquad \mu \in C^*(B). \tag{1}$$

Our main result may be formulated as follows.

THEOREM. *Put $R^m \setminus G = C$, suppose that $\mathscr{H}_m(C) > 0$ and denote by D that of the sets G, C which is bounded. Then the following assertions hold.*

I. *The operator*

$$T_G{}^*\colon \mu \mapsto T_G{}^*\mu \tag{2}$$

is nonexpansive on $C^*(B)$ (which means that

$$\sup\{\| T_G{}^*\mu \|; \mu \in C^*(B), \| \mu \| = 1\} \leqslant 1),$$

if and only if D is convex.

II. *If D is convex, then the operator $T_G{}^*$ is contractive on*

$$C_0{}^*(B) = \{\mu \in C^*(B); \mu(B) = 0\}$$

(in the sense that

$$\sup\{\| T_G{}^*\mu \|; \mu \in C_0{}^*(B), \| \mu \| = 1\} < 1),$$

if and only if $K_{z_1}(D) \cap K_{z_1}(D) \neq \bar{D}$ for every couple of points z_1, $z_2 \in B$.

III. *If D is convex, then the operator $(T_G{}^*)^2 = T_G{}^*(T_G{}^* \cdot)$ is always contractive on $C_0{}^*(B)$.*

Remark 1. Let us denote by δ_y the Dirac measure ($=$ unit point-mass) concentrated at $y \in R^m$. Fubini's theorem yields for any $\mu \in C^*(B)$ the formula

$$\langle \varphi, N^G U\mu \rangle = \int_B \langle \varphi, N^G U\delta_y \rangle \, d\mu(y), \qquad \varphi \in \mathscr{D}. \tag{3}$$

If $\mathscr{H}_m(C) = 0$, then

$$N^G U\delta_y = 2\delta_y \qquad (y \in B)$$

(see [5, p. 513 and footnote 2]) which together with (3) give $T_G{}^*\mu = \mu$; we see that, in this special case, the operator (2) reduces to the identity mapping.

If the operator (2) is bounded on $C^*(B)$, then necessarily

$$\sup_{y \in B} \| N^G U\delta_y \| < \infty. \tag{4}$$

For the proof of our theorem we need several auxiliary results concerning those G's obeying (4). Let us first recall some terminology and notation. Put

$\Omega_r(z) = \{x \in R^m; |x - z| < r\}$. If $\theta \in \Gamma$, let $\Omega_r(z, \theta) = \{x \in \Omega_r(z); (x - z) \cdot \theta > 0\}$. A vector $\theta \in \Gamma$ is termed the interior normal of G at z (in Federer's sense) if

$$\lim_{r \to 0+} \frac{\mathscr{H}_m(\Omega_r(z, \theta) \cap G)}{\mathscr{H}_m(\Omega_r(z))} = \frac{1}{2} = \lim_{r \to 0+} \frac{\mathscr{H}_m(\Omega_r(z, -\theta) \setminus G)}{\mathscr{H}_m(\Omega_r(z))} . \tag{5}$$

Such a vector $\theta \in \Gamma$, if it exists, is uniquely determined and is denoted by $n(z)$; if there is no $\theta \in \Gamma$ satisfying (5), we put $n(z) = 0$ (= zero vector in R^m). The set

$$\hat{B} = \{z \in R^m; |n(z)| = 1\}$$

is termed the reduced boundary of G; obviously, $\hat{B} \subset B$. Federer proved [2, Theorem 4.5] that the function $n: y \mapsto n(y)$ is Borel measurable on B.

LEMMA 1. *If* (4) *holds, then* $\mathscr{H}_{m-1}(\hat{B}) < \infty$; *for every* $z \in B$ *the integral*

$$\int_B \frac{n(y) \cdot (y - z)}{|y - z|^m} \, d\mathscr{H}_{m-1}(y)$$

converges, the density

$$d_G(z) = \lim_{r \to 0+} \frac{\mathscr{H}_m(\Omega_r(z) \cap G)}{\mathscr{H}_m(\Omega_r(z))}$$

exists and if we define the measure $\nu_z \in C^*(B)$ *on Borel sets* M *by*

$$\nu_z(M) = \int_{B \cap M} \frac{n(y) \cdot (y - z)}{|y - z|^m} \, d\mathscr{H}_{m-1}(y), \tag{6}$$

then

$$N^G U \delta_z = 2 d_G(z) \, \delta_z + (2/A) \, \nu_z . \tag{7}$$

Proof. This follows from results established in [5, Sect. 2, Lemma 3.2] (in view of a different normalizing of p,[2] the right-hand side of formula (3.3) must be multiplied by $2/A$).

LEMMA 2. *Let us assume* (4) *and denote for* $z \in B$ *and* $0 < r \leqslant \infty$ *by* $P_r(z)$ *the set of those* $\theta \in \Gamma$ *for which both* $\mathscr{H}_1(\Omega_r(z) \cap H_z(\theta) \cap G) > 0$ *and* $\mathscr{H}_1(\Omega_r(z) \cap H_z(\theta) \cap C) > 0$. *Then* $P_r(z)$ *is a Borel set and*

$$|\nu_z| \, (\Omega_r(z) \cap B) \geqslant \mathscr{H}_{m-1}(P_r(z)) \qquad (z \in B, 0 < r \leqslant \infty).$$

Proof. This follows from [5, Sect. 1.6 and 2].

We say that a vector $\theta \in \Gamma$ points into $M \subset R^m$ at $z \in R^m$ if there is an $\epsilon > 0$ such that

$$\mathscr{H}_1(\Omega_\epsilon(z) \cap H_z(\theta) \setminus M) = 0.$$

LEMMA 3. *Assume* (4), *fix* $z \in B$, *and denote by* K_z *the set of those* $\theta \in \Gamma$ *which point into* G *at* z. *Then* K_z *is measurable* ($\mathscr{H}_{m-1}$), $\mathscr{H}_{m-1}(K_z) = Ad_G(z)$ *and*

$$\begin{aligned} \nu_z(B) &= -\mathscr{H}_{m-1}(K_z), &\quad \text{if} \quad G = D, \\ &= A - \mathscr{H}_{m-1}(K_z), &\quad \text{if} \quad C = D. \end{aligned}$$

If $z \in \hat{B}$, *then* $\mathscr{H}_{m-1}$*-almost every* $\theta \in \Gamma$ *with* $\theta \cdot n(z) > 0$ *points into* G *at* z *and* $\mathscr{H}_{m-1}$*-almost every* $\theta \in \Gamma$ *with* $\theta \cdot n(z) < 0$ *points into* C *at* z.

Proof. See [5, Sects. 2.6, 2.7 and p. 535].

LEMMA 4. *If the operator* $T_G{}^*$ *is nonexpansive on* $C^*(B)$, *then* $| d_G(z) - \frac{1}{2} | < \frac{1}{2}$ *for every* $z \in B$.

Proof. Since both G and C have positive volume and one of these sets is bounded, we have $\mathscr{H}_{m-1}(P_\infty(z)) > 0$, whence it follows by Lemma 2

$$| \nu_z | (B) > 0, \qquad z \in B.$$

Since ν_z is nonatomic and

$$T_G{}^*\delta_z = [2d_G(z) - 1]\,\delta_z + (2/A)\,\nu_z \qquad (z \in B) \tag{8}$$

(see (1), (7)), we obtain

$$1 \geqslant \| T_G{}^*\delta_z \| = | 2d_G(z) - 1 | + (2/A) \| \nu_z \| > | 2d_G(z) - 1 | .$$

LEMMA 5. *If* $T_G{}^*$ *is nonexpansive on* $C^*(B)$, *then*

$$\mathscr{H}_{m-1}(\Omega_r(z) \cap \hat{B}) > 0$$

for every $z \in B$ *and* $r > 0$.

Proof. This follows at once from Lemma 4 and [5, Lemma 3.7].
Now we are in position to present the following

Proof of the Assertion I. Suppose first that $T_G{}^*$ is nonexpansive on $C^*(B)$. Consider $z \in \hat{B}$. By Lemma 3, one of the half-spheres

$$\{\theta \in \Gamma;\, \theta \cdot n(z) \geqslant 0\}, \qquad \{\theta \in \Gamma;\, \theta \cdot n(z) \leqslant 0\}$$

has the property that $\mathscr{H}_{m-1}$-almost every of its elements points into D at z; we denote by $\Gamma_D(z)$ this half-sphere and put

$$D(z) = \{z + t\theta;\, \theta \in \Gamma_D(z),\, t \geqslant 0\}.$$

We assert that

$$B \subset D(z). \tag{9}$$

In the opposite case Lemma 4 implies that $R^m \setminus D(z)$ meets D in a set of positive volume. Consequently, the set

$$\Gamma_1 = \{\theta \in \Gamma \setminus \Gamma_D(z);\ \mathscr{H}_1(H_z(\theta) \cap D) > 0\}$$

has positive $\mathscr{H}_{m-1}$-measure. Since D is bounded, we observe that

$$\Gamma_1 = P_\infty(z) \setminus \Gamma_D(z)$$

and $\mathscr{H}_{m-1}(\Gamma_D(z) \setminus P_\infty(z)) = 0$. This together with Lemma 2 gives

$$\begin{aligned} \| \nu_z \| &\geqslant \mathscr{H}_{m-1}(P_\infty(z)) = \mathscr{H}_{m-1}(P_\infty(z) \cap \Gamma_D(z)) + \mathscr{H}_{m-1}(P_\infty(z) \setminus \Gamma_D(z)) \\ &> \mathscr{H}_{m-1}(\Gamma_D(z)) = \tfrac{1}{2}A, \end{aligned}$$

whence $\| T_G{}^* \delta_z \| = (2/A) \| \nu_z \| > 1$ by (8). This contradiction proves (9) which in turn implies $\bar{D} \subset D(z)$ for any $z \in \hat{B}$. Put

$$D_1 = \bigcap_{z \in \hat{B}} D(z)$$

and denote by B_1 the boundary of D_1. Then D_1 is a closed convex set containing $\bar{D}$. Since $\hat{B} \subset B_1$ and $\hat{B}$ is dense in B by Lemma 5, we have also $B \subset B_1$ and, consequently, $\bar{D} = D_1$. In case $D = C$ the proof is complete. If $D = G$, then the inclusion $B \subset B_1$ shows that G must coincide with the interior of D_1 and convexity of G is established.

Conversely, suppose that D is convex. Then (4) holds by [5, 1.6 and 1.13], where a geometrical interpretation of (4) is given. From definition (1) and formula (3) we obtain for any $\mu \in C^*(B)$

$$\langle \varphi, T_G{}^* \mu \rangle = \int_B \langle \varphi, T_G{}^* \delta_y \rangle \, d\mu(y), \qquad \varphi \in \mathscr{D}. \tag{10}$$

In order to verify the implication

$$\mu \in C^*(B) \Rightarrow \| T_G{}^* \mu \| \leqslant \| \mu \|$$

it is therefore sufficient to show that

$$\| T_G{}^* \delta_z \| = 1, \qquad z \in B. \tag{11}$$

We distinguish the cases $D = G$ and $D = C$. Let $z \in B$. If $D = G$, then $d_G(z) \leqslant \frac{1}{2}$ and, by definition (6), $\nu_z(\cdot) \leqslant 0$. By (8) and Lemma 3 we obtain

$$\| T_G{}^* \delta_z \| = 1 - 2d_G(z) - (2/A)\, \nu_z(B) = 1.$$

If $D = C$, then $1 - d_G(z) \leqslant \frac{1}{2}$ and $\nu_z(\cdot) \geqslant 0$. Using (8) and Lemma 3 we now get

$$\| T_G{}^* \delta_z \| = 2d_G(z) - 1 + (2/A)\, \nu_z(B) = 1.$$

Thus (11) is verified and the proof of I is complete.

Remark 2. As we have noticed above, condition (4) is satisfied if D is convex. Assuming (4) we put for $z \in B$

$$\lambda_z = [2d_G(z) - 1]\,\delta_z + (2/A)\,\nu_z\,, \tag{12}$$

so that $\lambda_z = T_G{}^*\delta_z \in C^*(B)$ and

$$\sup_{z\in B} \|\lambda_z\| < \infty. \tag{13}$$

We denote by $C(B)$ the Banach space of all continuous real-valued functions on B (equipped with the supremum norm). Given $f \in C(B)$ we define

$$T_G f(z) = \int_B f\, d\lambda_z\,, \qquad z \in B. \tag{14}$$

It follows from [5, Sects. 3.1–3.4] that $T_G f \in C(B)$ whenever $f \in C(B)$. According to (13), the operator

$$T_G : f \mapsto T_G f \tag{15}$$

is bounded on $C(B)$. We have the duality between $C(B)$ and $C^*(B)$ given by

$$\langle f, \mu\rangle = \int_B f\, d\mu \qquad (f \in C(B),\ \mu \in C^*(B))$$

and it follows easily from (8), (10), (12), that

$$\langle f, T_G{}^*\mu\rangle = \langle T_G f, \mu\rangle \qquad (f \in C(B),\ \mu \in C^*(B)),$$

which shows that operator (2) is dual to (15). Let us agree to denote by $\mathbf{1}$ the function identically equal to 1 on B and put $\iota = -1$ or $\iota = 1$ according to whether G is bounded or not. Employing (12) and Lemma 3 we obtain

$$\iota T_G \mathbf{1} = \mathbf{1}. \tag{16}$$

The operator ιT_G is called the Neumann operator corresponding to G. According to (16) and assertion I, the following conditions (i)–(iv) are mutually equivalent:

(i) $\iota T_G \geqslant 0$ (which means that $\iota T_G f \geqslant 0$ for every nonnegative $f \in C(B)$;

(ii) $\| T_G \| \leqslant 1$;

(iii) $\| T_G{}^* \| \leqslant 1$;

(iv) D is convex.

An alternate proof of the equivalence (i) $\Leftrightarrow$ (iv) for $D = G$ is given in [8, Sect. 11].

If $Q \neq \varnothing$ is a compact subset of B, we put for $f \in C(B)$

$$\operatorname{osc} f(Q) = \max f(Q) - \min f(Q).$$

Then

$$\operatorname{osc} f(B) = 2 \sup\{\langle f, \mu \rangle; \mu \in C_0^*(B), \| \mu \| = 1\}$$

and the following formulas are easily verified:

$$\begin{aligned} &\sup\{\| T_G^* \mu \|; \mu \in C_0^*(B), \| \mu \| = 1\} \\ &\quad = \tfrac{1}{2} \sup\{\operatorname{osc} T_G f(B); f \in C(B), \| f \| \leqslant 1\} \\ &\quad = \tfrac{1}{2} \sup\{\| \lambda_{z_1} - \lambda_{z_2} \|; z_1, z_2 \in B\} \\ &\quad = \sup\{\operatorname{osc} T_G f(B); f \in C(B), \operatorname{osc} f(B) \leqslant 1\}. \end{aligned} \tag{17}$$

We now proceed with the proof of the assertions II, III. In the rest of this paper we always assume that D is convex. The interior of a set $M \subset R^m$ is denoted by int M.

LEMMA 6. *If $z_1, z_2 \in B$ and $K_{z_1}(D) \cap K_{z_1}(D) \neq \bar{D}$, then there are constants $q \in (0, 1)$ and $\rho > 0$ such that $\| \lambda_{x_1} - \lambda_{x_2} \| \leqslant 2q$ whenever $x_i \in B$, $| x_i - z_i | \leqslant \rho$ $(i = 1, 2)$.*

Proof. Choose $z \in B \cap \operatorname{int} K_{z_1}(D) \cap \operatorname{int} K_{z_2}(D)$ and fix $r > 0$ such that $\overline{\Omega_r(z)} \subset \operatorname{int} K_{z_1}(D) \cap \operatorname{int} K_{z_2}(D)$. Combining Lusin's theorem and Lemmas 1 and 5 we get a compact set $Q \subset \hat{B} \cap \overline{\Omega_r(z)}$ with $\mathscr{H}_{m-1}(Q) > 0$ such that the function $n: y \mapsto n(y)$ is continuous on Q. Next fix $\rho > 0$ such that $Q \cap \overline{\Omega_\rho(z_i)} = \varnothing$ and put $U_i = B \cap \overline{\Omega_\rho(z_i)}$ $(i = 1, 2)$. For every $y \in Q$ and every choice of $x_i \in U_i$ the segment $\{x_i + t(y - x_i); 0 < t < 1\}$ is contained in int D, so that the quantities

$$n(y) \cdot (y - x_1), \qquad n(y) \cdot (y - x_2)$$

are both different from zero and of the same sign. (Note that there is a unique supporting hyperplane to $\bar{D}$ at any $y \in \hat{B}$.) Hence

$$\begin{aligned} &\left| \frac{n(y) \cdot (y - x_1)}{| y - x_1 |^m} - \frac{n(y) \cdot (y - x_2)}{| y - x_2 |^m} \right| \\ &\qquad < \frac{| n(y) \cdot (y - x_1)|}{| y - x_1 |^m} + \frac{| n(y) \cdot (y - x_2)|}{| y - x_2 |^m}. \end{aligned} \tag{18}$$

Since the functions occurring in (18) are all continuous on the compact set $Q \times U_1 \times U_2$ $(y \in Q, x_i \in U_i)$, there is an $\epsilon > 0$ such that

$$\begin{aligned} &\left| \frac{n(y) \cdot (y - x_1)}{| y - x_1 |^m} - \frac{n(y) \cdot (y - x_2)}{| y - x_2 |^m} \right| + \epsilon \\ &\qquad \leqslant \frac{| n(y) \cdot (y - x_1)|}{| y - x_1 |^m} + \frac{| n(y) \cdot (y - x_2)|}{| y - x_2 |^m} \end{aligned}$$

on $Q \times U_1 \times U_2$. This estimate together with (6), (11), (12), gives

$$\begin{aligned}
&\| \lambda_{x_1} - \lambda_{x_2} \| \\
&\quad \leqslant | 2d_G(x_1) - 1 | + | 2d_G(x_2) - 1 | + (2/A) \| \nu_{x_1} - \nu_{x_2} \| \\
&\quad \leqslant | 2d_G(x_1) - 1 | + | 2d_G(x_2) - 1 | + 2A^{-1}(-\epsilon \mathscr{H}_{m-1}(Q) + \| \nu_{x_1} \| + \| \nu_{x_2} \|) \\
&\quad \leqslant -2A^{-1}\epsilon \mathscr{H}_{m-1}(Q) + \| \lambda_{x_1} \| + \| \lambda_{x_2} \| = 2(1 - \mathscr{H}_{m-1}(Q)\, \epsilon A^{-1})
\end{aligned}$$

whenever $x_i \in U_i$ ($i = 1, 2$) (compare [7, Sect. 4]).

COROLLARY 1. *Let P be a compact subset of B such that $K_{z_1}(D) \cap K_{z_2}(D) \neq \bar{D}$ whenever $z_1, z_2 \in P$. Then*

$$2 \sup\{\operatorname{osc} T_G f(P); f \in C(B), \operatorname{osc} f(B) \leqslant 1\} = \sup\{\| \lambda_{z_1} - \lambda_{z_2} \|; z_1, z_2 \in P\} < 2.$$

Proof. This follows at once from the compactness of $P \times P$ and Lemma 6. Now we may complete the

Proof of Assertion II. If $K_{z_1}(D) \cap K_{z_2}(D) \neq \bar{D}$ for every couple of points $z_1, z_2 \in B$, then $T_G{}^*$ is contractive on $C_0{}^*(B)$ by (17) and Corollary 1 (where we let $P = B$). Suppose now that $\bar{D} = K_{z_1}(D) \cap K_{z_2}(D)$ for suitable $z_1, z_2 \in B$. Since $\bar{D}$ is bounded, necessarily $z_1 \neq z_2$. Let us define the function g on B as follows (cf. [14, Remark 2]) $g(z_1) = 1$, $g = 1$ on $(B \backslash \{z_2\}) \cap \operatorname{int} K_{z_1}(D)$, $g = 0$ elsewhere on B. If $y \in B \backslash \{z_2\}$ and $g(y) = 0$, then either $| n(y) | = 0$ or else $n(y) \cdot (y - z_1) = 0$. This together with (6), (12) shows that $g = 1$ almost everywhere (λ_{z_1}) on B, whence by (16)

$$\int_B g \, d\lambda_{z_1} = \lambda_{z_1}(B) = T_G 1(z_1) = \iota. \tag{19}$$

On the other hand, if $y \in B \cap \operatorname{int} K_{z_1}(D)$, then necessarily y belongs to the boundary of $K_{z_2}(D)$ and $n(y) \cdot (y - z_2) = 0$. Hence we infer by (6), (12) that $g = 0$ almost everywhere (λ_{z_2}) and

$$\int_B g \, d\lambda_{z_2} = 0. \tag{20}$$

Suppose now that $T_G{}^*$ is contractive on $C_0{}^*(B)$. According to (17) there is a $q \in (0, 1)$ such that $\operatorname{osc} T_G f(B) \leqslant q$ for every $f \in C(B)$ with $\operatorname{osc} f(B) \leqslant 1$. Noting that g is of the first class of Baire, we can choose a sequence $f_n \in C(B)$ such that

$$0 \leqslant f_n \leqslant 1, \qquad f_n \to g \qquad \text{pointwise on } B \quad \text{as} \quad n \to \infty.$$

Hence

$\int_B g d\lambda_{z_i} = \lim_{n\to\infty} T_G f_n(z_i)$ ($i = 1, 2$) and the estimate $| T_G f_n(z_1) - T_G f_n(z_2) | \leqslant q$ contradicts (19), (20).

Now we are engaged with the operator

$$T_G^2: f \mapsto T_G(T_G f) \qquad (f \in C(B))$$

which is the second iterate of the Neumann operator.

LEMMA 7. *For every couple of points $z_1, z_2 \in B$ there are constants $q \in (0, 1)$, $\rho > 0$ such that $| T_G^2 f(x_1) - T_G^2 f(x_2)| \leqslant q \operatorname{osc} f(B)$ whenever $f \in C(B)$, $| x_i - z_i | \leqslant \rho$, $x_i \in B$ $(i = 1, 2)$.*

Proof. Since D is bounded, we have $B \cap \operatorname{int} K_{z_i}(D) \neq \varnothing$. Employing Lusin's theorem and Lemmas 1 and 5 we fix a compact set $Q_i \subset \hat{B} \cap \operatorname{int} K_{z_i}(D)$ with $\mathscr{H}_{m-1}(Q_i) > 0$ such that $n: y \mapsto n(y)$ is continuous on Q_i. Since $| n(y) \cdot (y - z_i)| > 0$ for $y \in Q_i$ we may choose $\rho_i > 0$ with $\overline{\Omega_{\rho_i}(z_i)} \cap Q_i = \varnothing$ and $\epsilon_i > 0$ such that

$$(y \in Q_i\,, x_i \in B, | x_i - z_i | \leqslant \rho_i) \Rightarrow | n(y) \cdot (y - x_i)|/| y - x_i |^m \geqslant \epsilon_i\,. \tag{21}$$

Put $P = Q_1 \cup Q_2$. Since $\bar{D}$ is bounded and $K_y(D)$ is a half-space for any $y \in P$, we have $K_{y_1}(D) \cap K_{y_2}(D) \neq \bar{D}$ for every couple of points $y_1, y_2 \in P$. By Corollary 1,

$$\sup\{\operatorname{osc} T_G f(P); f \in C(B), \operatorname{osc} f(B) \leqslant 1\} = q_0 < 1. \tag{22}$$

Consider now an arbitrary $f \in C(B)$ with $\operatorname{osc} f(B) \leqslant 1$. Since $\operatorname{osc} T_G f(P) \leqslant q_0$ and $T_G \mathbf{1} = \iota \mathbf{1}$, we may choose $c \in R^1$ such that for $f_0 = f + c\mathbf{1}$ the inequalities

$$\tfrac{1}{2}(1 - q_0) \leqslant T_G f_0 \leqslant \tfrac{1}{2}(1 + q_0) \tag{23}$$

hold on P. Writing $U_i = B \cap \overline{\Omega_{\rho_i}(z_i)}$ we get by (21), (23) the following estimates for any $x_i \in U_i$

$$\begin{aligned}
\iota T_G^2 f_0(x_i) &= \int_B \iota T_G f_0(y)\, d\lambda_{x_i}(y) \\
&= \left(\int_{B \setminus Q_i} + \int_{Q_i}\right) \iota T_G f_0(y)\, d\lambda_{x_i}(y) \\
&\leqslant \iota\lambda_{x_i}(B \setminus Q_i) + \iota\tfrac{1}{2}(1 + q_0)\, \lambda_{x_i}(Q_i) \\
&= \iota[\lambda_{x_i}(B) - ((1 - q_0)/2)\, \lambda_{x_i}(Q_i)] \\
&\leqslant 1 - \tfrac{1}{2}(1 - q_0)\, | \nu_{x_i}(Q_i)| \\
&\leqslant 1 - \tfrac{1}{2}(1 - q_0)\, \epsilon_i \mathscr{H}_{m-1}(Q_i),
\end{aligned}$$

$$\begin{aligned}
\iota T_G^2 f_0(x_i) &\geqslant \int_{Q_i} T_G f_0(y)\, \iota\, d\lambda_{x_i}(y) \\
&\geqslant \tfrac{1}{2}(1 - q_0)\, \iota\nu_{x_i}(Q_i) \\
&\geqslant \tfrac{1}{2}(1 - q_0)\, \epsilon_i \mathscr{H}_{m-1}(Q_i).
\end{aligned}$$

We see that the values of $\iota T_G{}^2 f_0$ on U_i are comprised between the limits q_i, $1 - q_i$, where

$$q_i = \tfrac{1}{2}(1 - q_0)\, \epsilon_i \mathscr{H}_{m-1}(Q_i).$$

Letting $\rho = \min(\rho_1, \rho_2)$, $q = 1 - q_1 - q_2$, we have thus for any choice of $x_i \in B$ with $| x_i - z_i | \leqslant \rho$ the estimate

$$| T_G{}^2 f(x_1) - T_G{}^2 f(x_2)| = | \iota T_G{}^2 f_0(x_1) - \iota T_G{}^2 f_0(x_2)| \leqslant q$$

and the proof is complete.

Proof of Assertion III. Lemma 7 combined with a standard compactness argument gives the inequality

$$\sup\{\operatorname{osc} T_G{}^2 f(B); f \in C(B), \operatorname{osc} f(B) \leqslant 1\} < 1$$

which together with the equality (cf. (17))

$$\sup\{\|(T_G{}^*)^2 \mu \|; \mu \in C_0{}^*(B), \| \mu \| = 1\} = \sup\{\operatorname{osc} T_G{}^2 f(B); f \in C(B), \operatorname{osc} f(B) \leqslant 1\}$$

completes the proof of III.

Remark 3. If

$$N^G U \colon \mu \mapsto N^G U\mu$$

denotes the operator sending $\mu \in C^*(B)$ into the generalized normal derivative of its Newtonian potential, then (cf. (1))

$$N^G U = I^* + T_G{}^*$$

where I^* stands for the identity operator on $C^*(B)$. Contractivity of $(T_G{}^*)^2$ shows that, on $C_0{}^*(B)$, the operator $I^* + T_G{}^*$ has the inverse $[I^* - (T_G{}^*)^2]^{-1}(I^* - T_G{}^*)$. This settles the generalized Neumann problem in the following formulation: Given $\nu \in C_0{}^*(B)$, determine a $\mu \in C_0{}^*(B)$ with $N^G U\mu = \nu$. There is always a uniquely determined solution which is given by the Neumann series

$$\mu = \nu + \sum_{n=1}^{\infty} [(T^*{}_G)^{2n}\, \nu - (T^*{}_G)^{2n-1}\, \nu]$$

convergent in the norm of $C_0{}^*(B)$.

A dual result holds for the Dirichlet problem. The measure ν_z given by (6) can be introduced for any $z \in R^m$. Given $f \in C(B)$, the double layer potential

$$Wf(z) = \int_B f \, d\nu_z$$

represents a harmonic function of the variable z on $R^m \setminus B$ which, at any $y \in B$, satisfies the relation

$$\tfrac{1}{2}A(I + T_G)f(y) = \lim Wf(z), \qquad z \to y, \quad z \in \operatorname{int} C,$$

where I denotes the identity operator on $C(B)$ (cf. (2.19) in [5]). We know that $T_G{}^2$ is contractive on the factor-space $C(B)/C_0$, where $C_0 = \{c1; c \in R^1\}$, so that $I + T_G$ has the inverse $[I - T_G{}^2]^{-1} (I - T_G)$ on the Banach space $C(B)/C_0$. Given $g \in C(B)$, we put $g_0 = 2A^{-1}g$ and the Neumann series

$$g_0 + \sum_{n=1}^{\infty} (T_G^{2n} g_0 - T_G^{2n-1} g_0)$$

provides an $f \in C(B)$ such that the function

$$y \mapsto \lim Wf(z) \qquad (z \to y, z \in \operatorname{int} C)$$

differs only by a constant function from g on B.

References

1. Yu. D. Burago and V. G. Mazja, "Potential Theory and Function Theory for Irregular Regions," Seminars in Mathematics, Vol. 3, V. A. Steklov Mathematical Institute, Leningrad, 1967, Consultants Bureau, New York, 1969.
2. H. Federer, The Gauss-Green theorem, *Trans. Amer. Math. Soc.* **58** (1945), 44–76.
3. O. D. Kellog, "Foundations of Potential Theory," Springer, Berlin, 1929 (reissued in 1953 by Dover Publications, New York).
4. R. E. Kleinman and W. L. Wendland, On Neumann's method for the exterior Neumann problem for the Helmholtz equation, *J. Math. Anal. Appl.* **57** (1977), 170–202.
5. J. Král, The Fredholm method in potential theory, *Trans. Amer. Math. Soc.* **125** (1966), 511–547.
6. H. Lebesgue, Sur la méthode de Carl Neumann, *J. Math. Pures Appl.*, 9e série, **16** (1937), 205–217, 421–423.
7. H. Lebesgue, "En marge du calcul des variations (Une introduction au calcul des variations et aux inégalités géométriques)", Monographie de l'Enseignement mathématique No. 12, Institut de mathématiques, Université de Génève, 1963.
8. I. Netuka, Fredholm radius of a potential theoretic operator for convex sets, *Časopis Pěst. Mat.* **100** (1975), 374–383.
9. C. Neumann, Zur Theorie des logarithmischen und des Newtonschen Potentials, Berichte über die Verhandlungen der Königlich Sachsischen Gesellschaft der Wissenschaften zu Leipzig **22** (1870), 49–56, 264–321.
10. C. Neumann, "Untersuchungen über das logarithmische und Newtonsche Potential," Teubner, Leipzig, 1877.
11. C. Neumann, "Über die Methode des arithmetischen Mittels," Hirzel, Leipzig, 1887 (erste Abhandlubg), 1888 (zweite Abhandlung).
12. J. Plemelj, "Potentialtheoretische Untersuchungen," Preisschriften der Fürstlich Jablonowskischen Gesellschaft zu Leipzig, Teubner, Leipzig, 1911.
13. F. Riesz and B. Sz. Nagy, "Leçons d'analyse fonctionnelle," Akadémiai Kiadó, Budapest, 1972.
14. G. Schober, Neumann's lemma, *Proc. Amer. Math. Soc.* **19** (1968), 306–311.